

# BRIDGE DECOMPOSITIONS WITH DISTANCES AT LEAST TWO

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ABSTRACT. For  $n$ -bridge decompositions of links in  $S^3$ , we propose a practical method to ensure that the Hempel distance is at least two.

## 1. INTRODUCTION

Hempel distance is a measure of complexity originally defined for Heegaard splittings of 3-manifolds [7]. The definition can be extended to bridge decompositions of links and it has been successfully applied to knot theory. For example, extending Hartshorn's [6] study for Heegaard splittings, Bachman-Schleimer [1] showed that the distance of a bridge decomposition of a knot bounds from below the genus of any essential surface in the knot exterior. Extending Scharlemann-Tomova's [13] for Heegaard splittings, Tomova [14] showed that the distance of a bridge decomposition bounds from below the bridge number of the knot or the Heegaard genus of the knot exterior.

However, it is difficult to calculate the Hempel distance of a general Heegaard splitting or bridge decomposition. While estimating it from above is a simple task in principle, it is a hard problem to estimate the distance from below.

For a Heegaard splitting, Casson-Gordon [4] introduced the rectangle condition to ensure that the distance is at least two. Lee [8] gave a weak version of rectangle condition which guarantees the distance to be at least one. Berge [2] gave a criterion for a genus two Heegaard splitting which guarantees the distance to be at least three. Lustig-Moriah [9] also gave a criterion to estimate the distance of a Heegaard splitting from below.

On the other hand, we could not find corresponding results for bridge decompositions in literature. In this paper, we observe that a bridge decomposition of a link in  $S^3$  can be described by a *bridge diagram*, and show that the *well-mixed condition* for a bridge diagram guarantees the distance to be at least two (see Section 3 for definitions). It may be regarded as a variation of the rectangle condition for Heegaard diagrams.

**Theorem 1.** *Suppose  $(T_+, T_-; P)$  is an  $n$ -bridge decomposition of a link in  $S^3$  for  $n \geq 3$ . If a bridge diagram of  $(T_+, T_-; P)$  satisfies the well-mixed condition, the Hempel distance  $d(T_+, T_-)$  is at least two.*

Recently, Masur-Schleimer [12] found an algorithm to calculate the Hempel distance of a Heegaard splitting with a bounded error term. The author imagine that their algorithm may also be applicable to bridge decompositions. However, the point

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of our result is its practicality: for any given bridge decomposition, we can easily obtain a bridge diagram and check whether it satisfies the well-mixed condition.

## 2. BRIDGE DECOMPOSITIONS AND THE HEMPEL DISTANCE

Suppose  $L$  is a link in  $S^3$  and  $P$  is a 2-sphere dividing  $S^3$  into two 3-balls  $B_+$  and  $B_-$ . Assume that  $L$  intersects  $P$  transversally and let  $\tau_\varepsilon$  be the intersection of  $L$  with  $B_\varepsilon$  for each  $\varepsilon = \pm$ . That is to say,  $(S^3, L)$  is decomposed into  $T_+ := (B_+, \tau_+)$  and  $T_- := (B_-, \tau_-)$  by  $P$ . We call the triple  $(T_+, T_-; P)$  an  $n$ -bridge decomposition of  $L$  if each  $T_\varepsilon$  is an  $n$ -string trivial tangle. Here,  $T_\varepsilon$  is called an  $n$ -string trivial tangle if  $\tau_\varepsilon$  consists of  $n$  arcs parallel to the boundary of  $B_\varepsilon$ . Obviously 1-bridge decompositions are possible only for the trivial knot, so we assume  $n \geq 2$  in this paper.

Consider a properly embedded disk  $D$  in  $B_\varepsilon$ . We call  $D$  an *essential disk* of  $T_\varepsilon$  if  $\partial D$  is essential in the surface  $\partial B_\varepsilon \setminus \tau_\varepsilon$  and  $D$  is disjoint from  $\tau_\varepsilon$ . Here, a simple closed curve on a surface is said to be *essential* if it neither bounds a disk nor is peripheral in the surface. Note that essential disks of  $T_+$  and  $T_-$  are bounded by some essential simple closed curves on the  $2n$ -punctured sphere  $P \setminus L$ .

The essential simple closed curves on  $P \setminus L$  form a 1-complex  $\mathcal{C}(P \setminus L)$ , called the *curve graph* of  $P \setminus L$ . The vertices of  $\mathcal{C}(P \setminus L)$  are the isotopy classes of essential simple closed curves on  $P \setminus L$  and a pair of vertices spans an edge of  $\mathcal{C}(P \setminus L)$  if the corresponding isotopy classes can be realized as disjoint curves. In the case of  $n = 2$ , this definition makes the curve graph a discrete set of points and so a slightly different definition is used.

The *Hempel distance* (or just the *distance*) of  $(T_+, T_-; P)$  is defined by

$$d(T_+, T_-) := \min\{d([\partial D_+], [\partial D_-]) \mid D_\varepsilon \text{ is an essential disk of } T_\varepsilon, (\varepsilon = \pm)\}$$

where  $d([\partial D_+], [\partial D_-])$  is the minimal distance between  $[\partial D_+]$  and  $[\partial D_-]$  measured in  $\mathcal{C}(P \setminus L)$  with the path metric. Because the curve graph is connected [10], the distance  $d(T_+, T_-)$  is a finite non-negative integer.

For 2-bridge decompositions, there is a unique essential disk for each of the 2-string trivial tangles. Moreover, the curve graph of a 4-punctured sphere is well understood (see Sections 1.5 and 2.1 in [11] for example) and so we can calculate the exact distance.

Suppose  $(T_+, T_-; P)$  is an  $n$ -bridge decomposition of a link  $L$  for  $n \geq 3$ . If  $d(T_+, T_-) = 0$ , there are essential disks  $D_+, D_-$  of  $T_+, T_-$ , respectively, such that  $[\partial D_+] = [\partial D_-]$ . We can assume  $\partial D_+ = \partial D_-$  indeed and so  $D_+ \cup D_-$  is a 2-sphere in  $S^3$ . Therefore,  $(T_+, T_-; P)$  is separated by the sphere into an  $m$ -bridge decomposition and an  $(n - m)$ -bridge decomposition of sublinks of  $L$ . By the definition of essential disks,  $m$  is more than 0 and less than  $n$ . Conversely, we can conclude that the distance is at least one if  $(T_+, T_-; P)$  is not a such one.

## 3. BRIDGE DIAGRAMS AND THE WELL-MIXED CONDITION

Suppose  $(T_+, T_-; P)$  is an  $n$ -bridge decomposition of a link  $L$  in  $S^3$  and  $T_+ = (B_+, \tau_+)$ ,  $T_- = (B_-, \tau_-)$ . For each  $\varepsilon = \pm$ , the  $n$  arcs of  $\tau_\varepsilon$  can be disjointly projected into  $P$ . Let  $p : L \rightarrow P$  be such a projection. A *bridge diagram* of  $(T_+, T_-; P)$  is a diagram of  $L$  obtained from  $p(\tau_+)$  and  $p(\tau_-)$ . In the terminology of [5],  $\tau_+, \tau_-$  are the overpasses and the underpasses of  $L$ .

Note that the boundary of a regular neighborhood of each arc of  $p(\tau_\varepsilon)$  in  $P$  bounds an essential disk of  $T_\varepsilon$  separating an arc of  $\tau_\varepsilon$ . In this sense a bridge diagram represents a family of essential disks of  $T_+, T_-$ . So we can think of it as something like a Heegaard diagram for a Heegaard splitting.

It is well known that a bridge decomposition is displayed as a “plat” as in Figure 1 (See [3]). Now we describe how to convert a plat presentation to a bridge diagram. For example, consider a 3-bridge decomposition with a plat presentation as in the left of Figure 2. Here  $P$  can be isotoped onto any height, so start with  $P$  in the position  $P_s$ . The top in the right of Figure 2 illustrates a view of a canonical projection of the arcs  $t_+^1, t_+^2, t_+^3$  on  $P$  from  $B_+$  side. In our pictures,  $p(t_+^1), p(t_+^2), p(t_+^3)$  are represented by a solid line, a dotted line, a broken line, respectively. Shifting  $P$  to the position  $P_1$ , the projections are as the second in the right of Figure 2. Shifting  $P$  further to the position  $P_2$ , the projections are as the third. By continuing this process, the projections are as in Figure 3 when  $P$  is in the position  $P_g$ . Then we can find a canonical projection of the arcs  $t_-^1, t_-^2, t_-^3$  and obtain a bridge diagram.

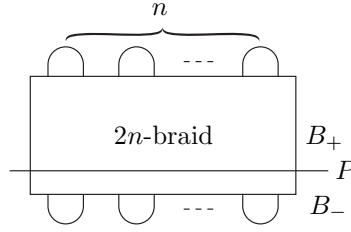


FIGURE 1.

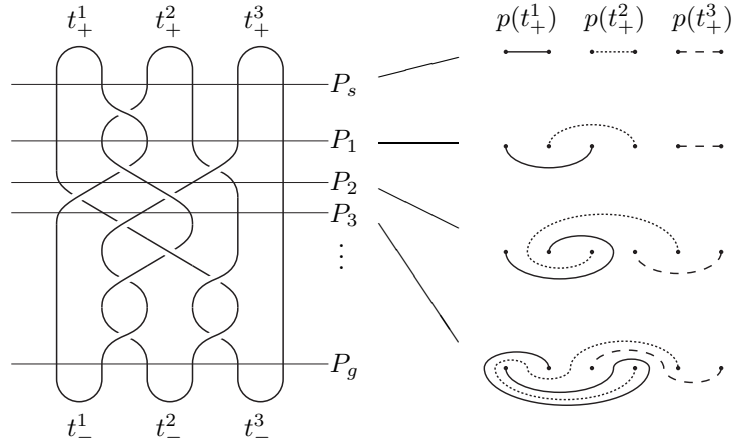


FIGURE 2.

Next we study the distance of this 3-bridge decomposition. Since the link  $L$  is connected, the bridge decomposition cannot be separated into smaller ones. It follows that the distance is at least one. Consider the simple closed curve  $c$  as in Figure 4. The curve  $c$  is essential in  $P \setminus L$  and disjoint from both  $p(t_+^1)$  and  $p(t_-^1)$ . Recall that the boundary of a small neighborhood of  $p(t_+^1), p(t_-^1)$  in  $P$  bounds an

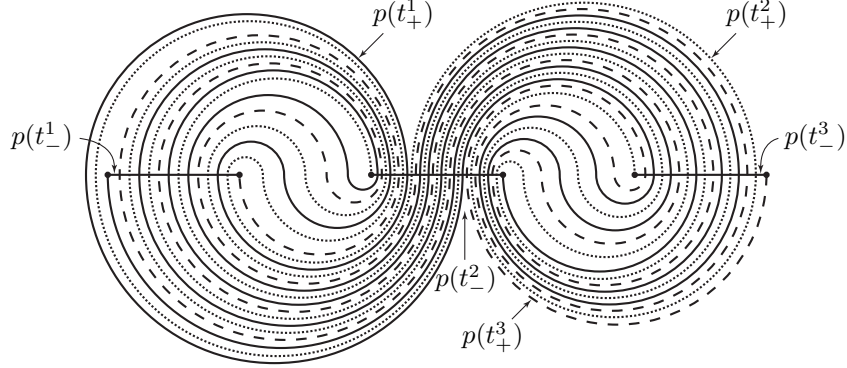


FIGURE 3.

essential disk  $D_+^1$  of  $T_+$  and an essential disk  $D_-^1$  of  $T_-$ , respectively. So there are an edge between  $[\partial D_+^1], [c]$  and an edge between  $[c], [\partial D_-^1]$  in the curve graph  $\mathcal{C}(P \setminus L)$ . By definition, the distance is at most two. It is true that there is no direct edge between  $[\partial D_+^1]$  and  $[\partial D_-^1]$ . However, this is not enough to conclude that the distance is equal to two because there are infinitely many essential disks of  $T_+, T_-$  other than  $D_+^1, D_-^1$ .

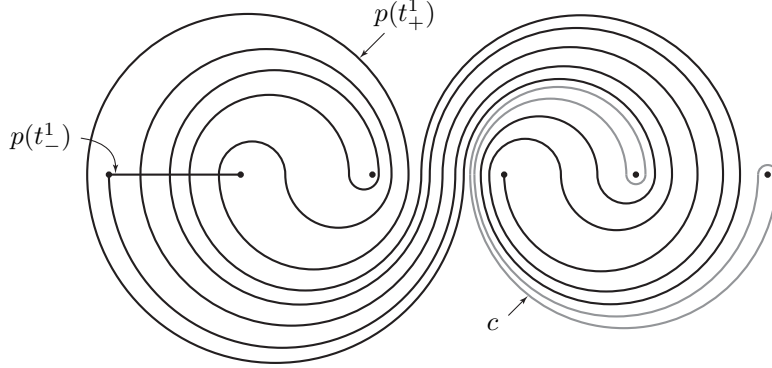


FIGURE 4.

As shown in [2], [4], [8] and [9], sufficiently complicated Heegaard diagram implies a large distance of the Heegaard splitting. We can expect that sufficiently complicated bridge diagram also implies a large distance of the bridge decomposition. A bridge diagram should be pretty complicated if it satisfies the *well-mixed condition*, which we define in the following.

Denote the arcs of each  $\tau_\varepsilon$  by  $t_\varepsilon^1, t_\varepsilon^2, \dots, t_\varepsilon^n$ . Let  $l$  be a loop on  $P$  containing  $p(\tau_-)$  such that  $p(t_-^1), p(t_-^2), \dots, p(t_-^n)$  are located in  $l$  in this order. We can assume that  $p(\tau_+)$  has been isotoped in  $P \setminus L$  to have minimal intersection with  $l$ . For the bridge diagram of Figure 3, it is natural to choose  $l$  to be the closure in  $P \cong S^2$  of the horizontal line containing  $p(t_-^1) \cup p(t_-^2) \cup p(t_-^3)$ . Let  $H_+, H_- \subset P$  be the hemi-spheres divided by  $l$  and let  $\delta_i$  ( $1 \leq i \leq n$ ) be the component of  $l \setminus p(\tau_-)$  which lies between  $p(t_-^i)$  and  $p(t_-^{i+1})$ . (Here the indices are considered modulo  $n$ .) Let  $\mathcal{A}_{i,j,\varepsilon}$  be the set of components of  $p(\tau_+) \cap H_\varepsilon$  separating  $\delta_i$  from  $\delta_j$  in  $H_\varepsilon$  for a distinct pair  $i, j \in \{1, 2, \dots, n\}$  and  $\varepsilon \in \{+, -\}$ . For example, Figure 5 displays

$\mathcal{A}_{1,2,+}$  for the above bridge diagram. Note that  $\mathcal{A}_{i,j,\varepsilon}$  consists of parallel arcs in  $H_\varepsilon$ .

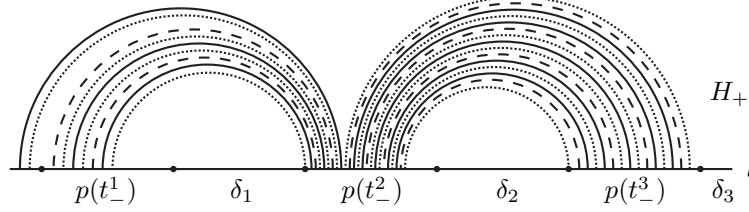


FIGURE 5.

- Definition 2.** (1) A bridge diagram satisfies the  $(i, j, \varepsilon)$ -well-mixed condition if in  $\mathcal{A}_{i,j,\varepsilon} \subset H_\varepsilon$ , a subarc of  $p(t_+^r)$  is adjacent to a subarc of  $p(t_+^s)$  for all distinct pair  $r, s \in \{1, 2, \dots, n\}$ .
- (2) A bridge diagram satisfies the *well-mixed condition* if it satisfies the  $(i, j, \varepsilon)$ -well-mixed condition for all combinations of a distinct pair  $i, j \in \{1, 2, \dots, n\}$  and  $\varepsilon \in \{+, -\}$ .

As in Figure 5, the bridge diagram in Figure 3 amply satisfies the  $(1, 2, +)$ -well-mixed condition. One can also check the  $(i, j, \varepsilon)$ -well-mixed condition for all the other combinations  $(i, j, \varepsilon) = (1, 2, -), (2, 3, +), (2, 3, -), (3, 1, +), (3, 1, -)$ . Hence the bridge diagram in Figure 3 satisfies the well-mixed condition.

#### 4. PROOF OF THE THEOREM

Firstly, consider an essential disk  $D_-$  of  $T_-$ . Assume that  $D_-$  has been isotoped so that  $|\partial D_- \cap l|$  is minimal. Here,  $|\cdot|$  denotes the number of connected components of a topological space.

**Lemma 3.** *There exist a distinct pair  $i, j \in \{1, 2, \dots, n\}$  and  $\varepsilon \in \{+, -\}$  such that  $\partial D_-$  includes a subarc connecting  $\delta_i$  and  $\delta_j$  in  $H_\varepsilon$ .*

*Proof.* Since the arcs of  $\tau_-$  are projected to subarcs of  $l$ , there exists a disk  $E_-$  in  $B_-$  such that  $\partial E_- = l$  and  $\tau_- \subset E_-$ . The essential disk  $D_-$  must have non-empty intersection with  $E_-$ . The closed components of  $D_- \cap E_-$  can be eliminated by an isotopy of  $\text{Int} D_-$ . Then  $D_- \cap E_-$  is a non-empty family of properly embedded arcs in  $D_-$ . Consider an outermost subdisk  $D_-^0$  of  $D_-$  cut off by an arc of them. For the minimality of  $|\partial D_- \cap l|$ , we can see that  $\partial D_-^0 \cap \partial D_-$  connects  $\delta_i$  and  $\delta_j$  in  $H_\varepsilon$  for a distinct pair  $i, j \in \{1, 2, \dots, n\}$  and  $\varepsilon \in \{+, -\}$ .  $\square$

Secondly, consider an essential disk  $D_+$  of  $T_+$ . Assume that  $D_+$  has been isotoped so that  $|\partial D_+ \cap p(\tau_+)|$  is minimal.

**Lemma 4.** *Suppose  $c$  is an essential simple closed curve on  $P \setminus L$  disjoint from  $\partial D_+$ . There exist a distinct pair  $r, s \in \{1, 2, \dots, n\}$  such that no subarc of  $c$  connects  $p(t_+^r)$  and  $p(t_+^s)$  directly (i.e. its interior is disjoint from  $p(\tau_+)$ ).*

*Proof.* Let  $E_+^i$  be a disk of parallelism between  $t_+^i$  and  $p(t_+^i)$  for each  $i = 1, 2, \dots, n$  so that  $E_+^1, E_+^2, \dots, E_+^n$  are pairwise disjoint. The closed components of  $D_+ \cap (E_+^1 \cup E_+^2 \cup \dots \cup E_+^n)$  can be eliminated by an isotopy of  $\text{Int} D_+$ . If  $D_+ \cap (E_+^1 \cup E_+^2 \cup \dots \cup E_+^n)$

is empty,  $D_+$  separates the  $n$  disks  $E_+^1, E_+^2, \dots, E_+^n$  into two classes in  $B_+$ . Since  $D_+$  is essential, both these classes are not empty. If  $D_+ \cap (E_+^1 \cup E_+^2 \cup \dots \cup E_+^n)$  is not empty, it consists of properly embedded arcs in  $D_+$ . Consider an outermost subdisk  $D_+^0$  of  $D_+$  cut off by an arc of them, say, an arc of  $D_+ \cap E_+^k$ . Then,  $D_+^0 \cup E_+^k$  separates the  $(n-1)$  disks  $E_+^1, \dots, E_+^{k-1}, E_+^{k+1}, \dots, E_+^n$  into two classes in  $B_+$ . Since  $|\partial D_+ \cap p(t_+^k)|$  is minimal, both these classes are not empty. Anyway, by choosing  $r$  and  $s$  from the indexes of the disks of separated classes, the lemma follows.  $\square$

Assume that the distance of  $(T_+, T_-; P)$  is less than two. There are disjoint essential disks  $D_+, D_-$  of  $T_+, T_-$ , respectively. If  $\partial D_-$  contains a subarc connecting  $\delta_i$  and  $\delta_j$  in  $H_\varepsilon$ , it intersects all the arcs of  $\mathcal{A}_{i,j,\varepsilon}$ . In particular, if two arcs of  $\mathcal{A}_{i,j,\varepsilon}$  are adjacent in  $H_\varepsilon$ , a subarc of  $\partial D_-$  connects them directly. The above observations and the well-mixed condition are almost enough to lead to a contradiction, but only the following should be checked:

**Lemma 5.** *The disks  $D_+$  and  $D_-$  can be isotoped preserving the disjointness so that  $|\partial D_+ \cap p(\tau_+)|$  and  $|\partial D_- \cap l|$  are minimal.*

*Proof.* Note that any isotopy of  $\partial D_\varepsilon$  in  $P \setminus L$  can be realized by an isotopy of  $D_\varepsilon$  in  $B_\varepsilon \setminus \tau_\varepsilon$  for  $\varepsilon = \pm$ .

If  $|\partial D_+ \cap p(\tau_+)|$  is not minimal, there are a subarc of  $\partial D_+$  and a subarc  $\alpha$  of  $p(\tau_+)$  cobounding a disk  $\Delta_+$  in  $P \setminus L$ . Since  $D_+, D_-$  are disjoint,  $\partial D_- \cap \Delta_+$  consists of arcs parallel into  $\alpha$ . Let  $\Delta_+^0$  be an outermost disk of the parallelisms. By assumption,  $p(\tau_+)$  has minimal intersection with  $l$  and so no component of  $l \cap \Delta_+^0$  has both end points on  $\alpha$ . By an isotopy of  $\partial D_-$  across  $\Delta_+^0$ , we can reduce  $|\partial D_- \cap \Delta_+|$  without increasing  $|\partial D_- \cap l|$ . After pushing out  $\partial D_-$  from  $\Delta_+$  in this way, we can reduce  $|\partial D_+ \cap p(\tau_+)|$  by an isotopy of  $\partial D_+$  across  $\Delta_+$ .

If  $|\partial D_- \cap l|$  is not minimal, there are a subarc of  $\partial D_-$  and a subarc  $\beta$  of  $l$  cobounding a disk  $\Delta_-$  in  $P \setminus L$ . The intersection  $\partial D_+ \cap \Delta_-$  consists of arcs parallel into  $\beta$ . Let  $\Delta_-^0$  be an outermost disk of the parallelisms. By the minimality of  $|l \cap p(\tau_+)|$ , no component of  $p(\tau_+) \cap \Delta_-^0$  has both end points at  $\beta$ . By an isotopy of  $\partial D_+$  across  $\Delta_-^0$ , we can reduce  $|\partial D_+ \cap \Delta_-|$  without increasing  $|\partial D_+ \cap p(\tau_+)|$ . After pushing out  $\partial D_+$  from  $\Delta_-$  in this way, we can reduce  $|\partial D_- \cap l|$  by an isotopy of  $\partial D_-$  across  $\Delta_-$ .  $\square$

Theorem 1 implies that the 3-bridge decomposition in Figure 2 has distance at least two. Since we have shown that it is at most two, the distance is exactly two. We can work out in this way fairly many  $n$ -bridge decompositions, especially for  $n = 3$ .

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